

Penalized Regression Models for the NBA

Dapo Omidiran *

Abstract

In the National Basketball Association (NBA), teams must make choices about which players to acquire, how much to pay them, and other decisions that are fundamentally dependent on player effectiveness. Thus, there is great interest in quantitatively understanding the impact of each player. In this paper we develop a new penalized regression model for the NBA, use cross-validation to select its tuning parameters, and then use it to produce ratings of player ability. We then apply the model to the 2010-2011 NBA season to predict the outcome of games. We compare the performance of our procedure to other known regression techniques for this problem, and demonstrate empirically that our model produces substantially better predictions. We evaluate the performance of our procedure against the Las Vegas gambling lines, and show that with a sufficiently large number of games to train on our model outperforms those lines. Finally, we demonstrate how the technique developed in this paper can be used to quantitatively identify “overrated” players who are less impactful than common wisdom might suggest.

Keywords: Basketball; Penalized Regression; Ridge Regression; Lasso; Convex Programming

Contents

1	Introduction	2
2	Notation	3
3	A Brief Introduction to the Game of Basketball	3
3.1	Statistical Modeling of Basketball	3
3.2	Least Squares Estimation	5
3.3	Is least squares regression a good estimator of player value?	6
4	SPR: Improving Least Squares	8
4.1	Bayesian Interpretation of SPR	10
4.2	Selecting the regularization parameter $\tilde{\lambda}$	10

¹Dapo Omidiran is at the University of California, Berkeley, CA 94720 (e-mail: dapo@eecs.berkeley.edu).

5	The Performance of SPR	11
5.1	SPR outperforms least squares	11
5.2	SPR outperforms Las Vegas	12
5.3	Robustness of Results	13
6	What does SPR say about the NBA?	13
6.1	Top 10 players in the league	13
6.2	Top 10 most underrated and overrated players	15
6.3	Box score weights produced by SPR	15
7	Extending SPR by augmenting the box score	16
8	Conclusion	17
A	The Cyclical Coordinate Descent Algorithm for SPR	19
A.1	Convergence of Algorithm 1	19
A.2	Computing the updates for Algorithm 1	20
A.2.1	Updates for β	20
A.2.2	Proof of Lemma A.2	22

1 Introduction

The National Basketball Association (NBA) is a multi-billion dollar business. Each of the thirty franchises in the NBA try their best to put forward the most competitive team possible within their budget. To accomplish this goal, a key task is to understand how good players are.

A large fraction of the thirty NBA teams have quantitative groups analyzing data to evaluate and rate players. The website ESPN.com has many analysts providing statistical analysis for casual fans. Gambling houses use quantitative analysis to price bets on games, while gamblers try to use quantitative analysis to find attractive wagers.

A popular technique for producing player ratings is weighted least-squares (LS) regression¹. However, as we show later show, least squares is an approach with many flaws.

In this paper, we introduce a new penalized regression technique for estimating player ratings which we call Subspace Prior Regression (henceforth, SPR). SPR corrects some of the flaws of least squares for this problem setting, and has substantially better out-of-sample predictive performance. Furthermore, given sufficient training data SPR outperforms the Las Vegas wagering lines.

We interpret the ratings produced by SPR, discussing it identifies as the best players in the NBA (Section 6.1), who are the most overrated and underrated players (Section 6.2), and what SPR suggests is the relative importance of different basic actions within the game

¹this technique is also known as Adjusted Plus/Minus (APM) in the quantitative basketball community.

like three point shooting and turnovers (Section 6.3). Finally, we discuss some possible improvements to this model (Section 7).

2 Notation

We use the notation \mathbb{R}_+^d to indicate the set $\{x \in \mathbb{R}^d \mid x_i > 0 \forall i\}$. Let $\mathbf{1}_n \in \mathbb{R}^{n \times 1}$ denote the column vector of ones and $\mathbf{e}_i \in \mathbb{R}^{n \times 1}$ signify the i^{th} standard basis vector. We use $\mathbf{I}_p \in \mathbb{R}^{p \times p}$ to denote the identity matrix of size p , and $\mathbf{Diag}(\mathbf{w})$ to stand for a diagonal matrix with entries given by the vector \mathbf{w} . Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}_+^n$ we define the inner product as $\mathbf{a}^T \mathbf{b} := \sum_{i=1}^n a_i b_i$, the ℓ_p norm $\|\mathbf{a}\|_p := [\sum_{i=1}^n a_i^p]^{\frac{1}{p}}$ and finally the \mathbf{c} -weighted ℓ_p norm as $\|\mathbf{a}\|_{p,\mathbf{c}} := [\sum_{i=1}^n c_i a_i^p]^{\frac{1}{p}}$.

3 A Brief Introduction to the Game of Basketball

Each of the thirty teams in the NBA plays 82 games in a season, where 41 of these games are at their home arena and 41 are played away. Thus, there are 1,230 total games in an NBA regular season. Each team has a roster of roughly twelve to fifteen players. Games are usually 48 minutes long, and each of the two competing teams has exactly five players on the floor at a time. Thus, there are ten players on the floor for the duration of the game. Associated with each game is a box score, which records the statistics of the players who played in that game. Figure 1 contains a sample box score from an NBA game played on February 2nd, 2011 by the Dallas Mavericks (the home team) against the New York Knicks (the away team). Note that we only display the box score for the Mavericks players. Observe that there are 12 players listed in the box score, but only 11 who actually played for the Mavericks in this game. Each of the columns of this box score corresponds to a basic statistic of interest (the column REB in the box score denotes rebounds, AST denotes steals, etc.)

3.1 Statistical Modeling of Basketball

To statistically model the NBA, we must first extract from each game a dataset suitable for quantitative analysis. There is a standard procedure for this currently used by many basketball analysts (Kubatko et al., 2007; Oliver, 2004), which we describe as follows.

We model each basketball game as a sequence of n distinct events between two teams. During event i the home team scores Y_i more points than the away team. We use the variable p to denote the total number of players in the league (in a typical NBA season, $p \approx 450$.) We can then represent the current players on the floor for event i with a vector $\mathbf{X}_i \in \mathbb{R}^p$ defined as

Figure 1: Sample single-game boxscore for the Dallas Mavericks

 Dallas Mavericks														
STARTERS	MIN	FGM-A	3PM-A	FTM-A	OREB	DREB	REB	AST	STL	BLK	TO	PF	+/-	PTS
Brian Cardinal, PF	10	1-1	1-1	0-0	0	0	0	3	0	0	0	1	-3	3
Dirk Nowitzki, PF	33	10-16	2-4	7-7	1	10	11	3	0	1	1	2	+27	29
Tyson Chandler, C	29	6-9	0-0	3-4	3	8	11	0	0	1	1	3	+16	15
Jason Kidd, PG	32	2-10	2-7	0-0	1	5	6	10	1	0	1	1	+16	6
DeShawn Stevenson, SG	31	5-9	3-6	0-0	2	3	5	1	0	0	1	1	+9	13
BENCH	MIN	FGM-A	3PM-A	FTM-A	OREB	DREB	REB	AST	STL	BLK	TO	PF	+/-	PTS
Jason Terry, SG	30	6-12	0-3	0-1	1	3	4	2	1	0	2	2	+7	12
Shawn Marion, SF	23	1-8	0-0	3-4	1	9	10	2	0	0	4	3	-9	5
Brendan Haywood, C	16	2-4	0-0	0-0	0	2	2	0	0	1	0	4	+3	4
Jose Juan Barea, PG	30	7-12	3-4	5-5	1	2	3	3	1	0	3	0	+21	22
Ian Mahinmi, C	4	1-1	0-0	0-0	0	1	1	0	0	1	1	1	-3	2
Dominique Jones, SG	3	1-2	0-0	0-0	0	1	1	0	0	0	1	0	-4	2
Peja Stojakovic, SF	DNP NOT WITH TEAM													
TOTALS		FGM-A	3PM-A	FTM-A	OREB	DREB	REB	AST	STL	BLK	TO	PF		PTS
		42-84	11-25	18-21	10	44	54	24	3	4	15	18		113
		50.0%	44.0%	85.7%										
+/- denotes team's net points while the player is on the court.													Fast break points: 15 Points in the paint: 38 Team TO (points off): 16 (16)	

$$\mathbf{X}_{ij} = \begin{cases} 1 & \text{Player } j \text{ is on the floor for the home team} \\ -1 & \text{Player } j \text{ is on the floor for the away team} \\ 0 & \text{otherwise.} \end{cases}$$

Associated with event i is a weighting factor w_i . Roughly speaking, the i^{th} event happens for w_i minutes.

Figure 1 contains a sample box score. We summarize box score data like that of Figure 1 with the matrix $\mathbf{R}_{\text{Mavericks,Game } 1}$ which looks like

$$\mathbf{R}_{\text{Mavericks Game \#1}} = \begin{matrix} & \text{MIN} & \text{FGM} & \text{FGA} & \dots & \text{PTS} \\ \begin{matrix} \text{Brian Cardinal} \\ \text{Dirk Nowitzki} \\ \vdots \\ \text{Peja Stojakovic} \end{matrix} & \begin{pmatrix} 10 & 1 & 1 & \dots & 3 \\ 33 & 10 & 16 & \dots & 29 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \end{matrix}$$

This matrix records the statistics of the 12 players on the Dallas Mavericks roster for that particular game. If there are d basic statistics of interest in this box score, then $\mathbf{R}_{\text{Mavericks Game \#1}}^{\text{Mavericks}}$ is a matrix of size 12 by d .

One can imagine computing the aggregate box score matrix

$$\mathbf{R}^{\text{Mavericks}} = \sum_{t=1}^{82} \mathbf{R}_{\text{Game \#}t}^{\text{Mavericks}}$$

that summarizes the total statistics of these 12 players for an entire season. Finally, define the $p \times d$ matrix \mathbf{R} that vertically concatenates \mathbf{R}_j across the 30 teams in the NBA:

$$\mathbf{R} := \begin{matrix} \text{Team 1} \\ \text{Team 2} \\ \vdots \\ \text{Team 30} \end{matrix} \begin{pmatrix} \mathbf{R}^{\text{Mavericks}} \\ \mathbf{R}^{\text{Bulls}} \\ \vdots \\ \mathbf{R}^{\text{Celtics}} \end{pmatrix}.$$

\mathbf{R} summarizes the season box score statistics for all p players who played in the NBA for that year.

3.2 Least Squares Estimation

We want to determine the relationship between \mathbf{X}_i and \mathbf{Y}_i , i.e., find a function f such that $\mathbf{Y}_i \approx f(\mathbf{X}_i)$. One natural way to do this is through a linear regression model, which assumes that

$$\mathbf{Y}_i = \alpha_{\text{hca}}^* + \mathbf{X}_i^T \boldsymbol{\beta}^* + e_i, i = 1, 2, \dots, n.$$

Recall that the event i has a weighting factor w_i associated with it. Roughly speaking, event i happens for w_i minutes.

The scalar variable α_{hca}^* represents a home court advantage term, while the variable $\boldsymbol{\beta}^* \in \mathbb{R}^p$ is interpreted as the number of points each of the p players in the league “produces” per minute. This model recognizes players for whom their team is more effective because of their presence on the floor.

For notational convenience, we stack the variables \mathbf{Y}_i , w_i , and e_i into the n vectors \mathbf{Y} , \mathbf{W} , and \mathbf{E} and the variables \mathbf{X}_i into the $n \times p$ matrix \mathbf{X} . This yields the matrix expression

$$\mathbf{Y} = \mathbf{1}_n \alpha_{\text{hca}}^* + \mathbf{X} \boldsymbol{\beta}^* + \mathbf{E}. \quad (1)$$

Given observations (\mathbf{Y}, \mathbf{X}) and weights \mathbf{W} , we define the \mathbf{W} -weighted quadratic loss function as

$$L_{\text{quadratic}}(\alpha_{\text{hca}}, \boldsymbol{\beta}) := \frac{1}{\sum_{i=1}^n w_i} \|\mathbf{Y} - \mathbf{1}_n \alpha_{\text{hca}} - \mathbf{X} \boldsymbol{\beta}\|_{\mathbf{W}}^2. \quad (2)$$

A natural technique for estimating the variables α_{hca}^* and $\boldsymbol{\beta}^*$ is to minimize (2), i.e.,

$$(\hat{\alpha}_{\text{hca}}^{\text{LS}}, \hat{\boldsymbol{\beta}}^{\text{LS}}) = \arg \min_{\alpha_{\text{hca}}, \boldsymbol{\beta}} L_{\text{quadratic}}(\alpha_{\text{hca}}, \boldsymbol{\beta}), \quad (3)$$

resulting in a weighted least squares (LS) problem. The values $\hat{\boldsymbol{\beta}}^{\text{LS}}$ are known in the quantitative basketball community as the *adjusted plus/minus ratings*². The website Basketballvalue³ has computed $\hat{\boldsymbol{\beta}}^{\text{LS}}$ for several recent seasons.

²<http://www.82games.com/ilardi1.htm>

³<http://www.basketballvalue.com>

3.3 Is least squares regression a good estimator of player value?

Table 1: LS Player Ratings

Rank	Player	$\hat{\beta}_i^{\text{LS}}$
1	James, LeBron	12.62
2	Durant, Kevin	11.5
3	Nash, Steve	11.39
4	Paul, Chris	10.42
5	Nowitzki, Dirk	10.33
6	Collison, Nick	9.87
7	Wade, Dwyane	9.59
8	Hilario, Nene	8.56
9	Deng, Luol	8.46
10	Howard, Dwight	8.13

Table 1 lists the top ten players in the NBA for the combined 2009-2010 and 2010-2011 NBA regular seasons by their ratings produced from least squares⁴. By this ranking, LeBron James was the best player in the league over this two year period. Since $\hat{\beta}_{\text{LeBron James}}^{\text{LS}} = 12.62$, this procedure suggests that he is worth an additional 12.62 net points to his team for every 100 possessions the team plays.

How believable are the player ratings of Table 1? The list has many of the widely-considered best players in the NBA. However, there are also some names on this list that are questionable. If we believe these ratings, then Nick Collison, a player considered by most fans and analysts to be at best a merely average player at his position, is better than Dwyane Wade and Dwight Howard, two of the premiere superstars in the league. Similarly, while Nene Hilario and Luol Deng are good players, they are not considered by most fans and analysts to be amongst the top ten players in the NBA.

This contradiction between common wisdom and least squares is useful, since it can either reveal to us that the common wisdom is wrong or that the least squares approach is incorrect. We need some basis of comparison to evaluate how well least squares is performing.

In classical linear regression, assuming that the generative model satisfies certain conditions, the least squares estimate has several desirable properties (maximum likelihood estimate, best linear unbiased estimate, consistency, asymptotic normality, etc). However, these properties typically assume that the underlying model satisfies certain technical conditions like normality, linearity, and statistical independence. It is unreasonable to expect that these technical conditions hold for the game of basketball. Thus, we must find other ways to evaluate how trustworthy the $\hat{\beta}^{\text{LS}}$ values are, and whether they should be believed over common wisdom about players. One simple approach for evaluating the the least squares model is to test its predictive power versus a simple dummy estimator.

⁴These numbers were obtained from <http://basketballvalue.com/topplayers.php?&year=2010-2011>

To do this, we

1. define a dummy estimator that sets $\hat{\beta}_i^{\text{Dummy}} = 0$ for each player, and the home court advantage term $\hat{\beta}_{\text{hca}}^{\text{Dummy}} = 3.5$. In other words, each player is rated a zero, and the home team is predicted to win every 100 possessions by 3.5 points.
2. We then can compute both the least squares estimate and dummy estimate for the first 820 games of an NBA season, and measure how well each technique does in estimating the margin of victory of the home team for the remaining 410 games of that season.

If least squares accurately models the NBA, then at a minimum it must substantially outperform the dummy estimator. Let us use the variable A_k to denote the *actual* number of points by which the home team wins game k , \hat{A}_k to denote the *predicted* number of points by the statistical estimator of interest, and $\hat{E}_k := \hat{A}_k - A_k$ to denote the error this statistical estimator makes in predicting the outcome of game k .

Figure 2 is a histogram of the error variable \hat{E}_k over the course of the 410 games under consideration from the 2010-2011 NBA season for each technique. A perfect estimator would have a spike of height 410 centered around zero. Thus, the “spikier” the histogram looks, the better a method performs. It is hard to immediately say from Figure 2 that the least squares estimate yields better predictions than the simple dummy estimate. We can also study some of the empirical properties of \hat{E}_k for each approach. Table 2 summarizes the results.

Table 2: Performance of Statistical Estimators over the last 410 games

Metric	Dummy	LS	RR	SPR	SPR2
RMSE (millions)	547.0447	558.0350	551.0912	539.7495	541.9725
Fraction of games guessed wrong	0.3927	0.3366	0.3293	0.2854	0.2951
Mean of $ \hat{E}_i $	10.5394	18.0507	16.0016	10.5540	11.8109
Variance of $ \hat{E}_i $	68.5292	187.1446	147.8993	60.2527	70.7671
Median of $ \hat{E}_i $	9.4885	15.2577	13.6127	8.9687	10.2200
Min of $ \hat{E}_i $	0.0279	0.0346	0.0109	0.0260	0.1445
Max of $ \hat{E}_i $	38.8338	79.1965	72.0122	40.9012	45.9275
Empirical $\mathbb{P}(\hat{E}_i > 1)$	0.9171	0.9707	0.9683	0.9463	0.9512
Empirical $\mathbb{P}(\hat{E}_i > 3)$	0.7683	0.9000	0.8878	0.8171	0.8732
Empirical $\mathbb{P}(\hat{E}_i > 5)$	0.6707	0.8366	0.8122	0.7244	0.7780
Empirical $\mathbb{P}(\hat{E}_i > 10)$	0.4854	0.6585	0.6049	0.4415	0.5098

When comparing least squares to the dummy estimate, we notice that

1. least squares reduces the percentage of games in which the wrong winner is identified from 39.27% to 33.66% over the block of 410 games of interest.

- Unfortunately, the empirical behavior of \hat{E}_k seems to be substantially worse for least squares. For example, the empirical mean of $|\hat{E}_i|$ is 18.05 for least squares, while only 10.54 for the dummy estimator. Thus, least squares makes larger average errors when predicting the final margin of victory of games.

As a result, it is hard to convincingly argue that least squares approach is a better model for the NBA than the dummy estimate.

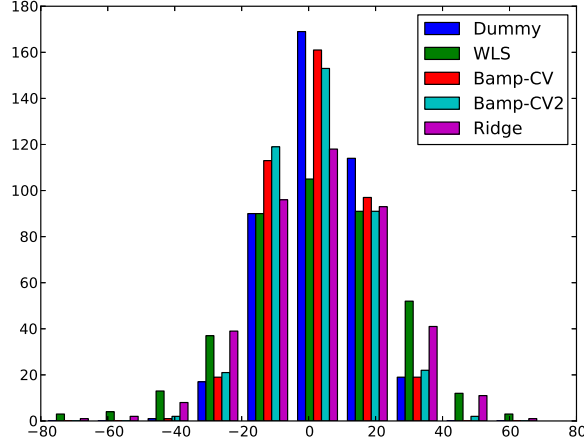


Figure 2. Comparison of Dummy, Least Squares, Ridge Regression, SPR and SPR2 trained on 820 games.

4 SPR: Improving Least Squares

Although Figure 2 and Table 2 suggest that the LS estimate performs poorly, this doesn't necessarily mean that the linear model (1) is without promise. The least squares estimate simply doesn't take into account the following two key pieces of information we have about the problem domain:

- Model sparsity: The NBA is a game dominated by star players. Lesser players have far less impact on wins and losses. This folk wisdom informs player acquisitions and salaries. For example, with a \$60 million budget, one would much rather acquire three elite \$15 million stars and fill out the rest of the roster with cheap role-players, than spend tons of money on role-players and skimp on stars.

This “elites first” strategy was used by the Boston Celtics in the summer of 2007 when they traded their role-players and other assets to build a team around Kevin Garnett, Paul Pierce and Ray Allen⁵, and more recently by the Miami Heat in the summer of

⁵<http://www.nba.com/celtics/news/press073107-garnett.html>

2010 who built a team around LeBron James, Dywane Wade and Chris Bosh⁶. We shall incorporate this prior information through ℓ_1 regularization. This penalizes non-sparse models, and should cause only the very best players to stand out in the regression. This suggests a penalty term of the form $\lambda_1 \|\beta\|_1$.

2. Box score information: Another valuable piece of information useful in inferring player worth is the box score statistics matrix \mathbf{R} . One expects good players to not only have high APM ratings, but to also produce rebounds, assists, blocks, steals, etc. Thus, we prefer ratings $\hat{\beta}$ which are consistent with box score statistics. In other words, we expect a ratings vector to be “close” to the column space of \mathbf{R} . We therefore should penalize ratings for which the distance from $\hat{\beta}$ to $\mathbf{R}\mathbf{z}$ is large. Although there are many different possible penalties one can choose, in this work we choose a quadratic penalty term of the form $\lambda_2 \|\beta - z_0 \mathbf{1}_p - \mathbf{R}\mathbf{z}\|_2^2$.

We can encode the above prior information through the function $g(\alpha_{\text{hca}}, \beta, z_0, \mathbf{z}; \vec{\lambda})$ defined as

$$g(\alpha_{\text{hca}}, \beta, z_0, \mathbf{z}; \vec{\lambda}) := \underbrace{L_{\text{quadratic}}(\alpha_{\text{hca}}, \beta)}_{\text{Weighted least squares}} + \underbrace{\lambda_1 \|\beta\|_1}_{\text{Sparse player ratings}} + \underbrace{\lambda_2 \|\beta - z_0 \mathbf{1}_p - \mathbf{R}\mathbf{z}\|_2^2}_{\text{Box score prior}}, \quad (4)$$

1. \mathbf{R} is a $p \times d$ matrix containing the box-score statistics of the p different players,
2. The variable \mathbf{z} gives us weights for each of the box score statistics,
3. and the vector $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ are the regularization parameters.

We shall use the shorthand $\vec{\lambda}$ to denote the pair (λ_1, λ_2) . We can find a model consistent with both the data and the prior information by solving the convex optimization problem

$$\hat{\beta}_{\text{hca}}^{\vec{\lambda}}, \hat{\beta}^{\vec{\lambda}}, \hat{\beta}_0^{\vec{\lambda}}, \hat{\mathbf{z}}^{\vec{\lambda}} = \arg \min g(\alpha_{\text{hca}}, \beta, z_0, \mathbf{z}; \vec{\lambda}). \quad (5)$$

We call the procedure described by Equation (5) the SPR algorithm, and the vector $\hat{\beta}^{\vec{\lambda}}$ are the player ratings produced by it. One very important difference between SPR and the least squares approach is that it yields both a player rating vector $\hat{\beta}^{\vec{\lambda}}$ and a box score weights vector $\hat{\mathbf{z}}^{\vec{\lambda}}$. The weights vector $\hat{\mathbf{z}}^{\vec{\lambda}}$ is a valuable tool in its own right. It provides numerical values for different basic box score statistics like scoring, rebounding, and steals. We further interpret $\hat{\mathbf{z}}^{\vec{\lambda}}$ in Section 6.3.

Furthermore, it yields a linear formula for transforming player box score effectiveness into a player productivity rating through the equation

$$\hat{\theta}^{\vec{\lambda}} := \mathbf{R}\hat{\mathbf{z}}^{\vec{\lambda}} + \hat{\beta}_0^{\vec{\lambda}} \mathbf{1}_p. \quad (6)$$

⁶<http://sports.espn.go.com/nba/news/story?id=5365165>

$\hat{\boldsymbol{\theta}}^{\vec{\lambda}}$ can be viewed as an additional player rating vector produced by SPR, one that linearly transforms each player’s box score production into a “points per 100 possession” rating similar to least squares or $\hat{\boldsymbol{\beta}}^{\vec{\lambda}}$. Thus, $\hat{\boldsymbol{\theta}}^{\vec{\lambda}}$ succinctly converts the box score production of each player into a single number.

Thus for player i , we can compare the variable $\hat{\beta}_i^{\vec{\lambda}}$ to the variable $\hat{\theta}_i^{\vec{\lambda}}$ to understand how “overrated” or underrated he is relative to his box score production. This is useful, since many players produce great box score statistics but don’t necessarily impact team competitiveness to the level the box score might suggest. We explore this aspect of SPR in further detail in Section 6.2.

4.1 Bayesian Interpretation of SPR

SPR can be interpreted as the posterior mode for a Bayesian statistics model. Suppose that $\mathbf{Y}, \alpha_{\text{hca}}, \boldsymbol{\beta}, z_0, \mathbf{z}$ are all random variables.

Let

- $\mathbf{Y}_i | \alpha_{\text{hca}}, \boldsymbol{\beta} \sim \mathcal{N}(\alpha_{\text{hca}} + \mathbf{X}_i^T \boldsymbol{\beta}, \frac{1}{2}),$
- α_{hca} have the improper prior $\mathbb{P}(\alpha_{\text{hca}} = \alpha) \propto 1$
- $\mathbb{P}(\boldsymbol{\beta} | z_0, \mathbf{z}) \propto e^{-\lambda_1 \|\boldsymbol{\beta}\|_1 - \lambda_2 \|\boldsymbol{\beta} - z_0 \mathbf{1}_p - \mathbf{Rz}\|_2^2},$
- z_0 have the improper prior $\mathbb{P}(z_0 = \gamma) \propto 1,$
- and \mathbf{z} has the improper prior $\mathbb{P}(\mathbf{z} = \kappa) \propto 1.$

Then the solution to SPR with $\mathbf{w} = \mathbf{1}_n$ is exactly the mode of the posterior distribution $\mathbb{P}(\alpha_{\text{hca}}, \boldsymbol{\beta}, z_0, \mathbf{z} | \mathbf{Y})$.

4.2 Selecting the regularization parameter $\vec{\lambda}$

For SPR to be useful, we need to be able to select a good choice of $\vec{\lambda}$ quickly. Cross-validation (Stone, 1974) is one standard technique in statistics for doing this. To select regularization parameters, we use 10-fold cross-validation. We cross-validate over regularization parameters from the set

$$\Lambda := \{(2^a, 2^b) \mid a, b \in F\}$$

where

$$F := \{-10, -9, \dots, 9\}.$$

K -fold cross-validation on T different values of $\vec{\lambda}$ means solving TK different SPR problems, each of which are convex programs of moderate size ($n \approx 20000$, $p \approx 450$, $d \approx 20$).

Thus, it is necessary that

Table 3: Regularization parameters obtained from 10-fold cross-validation

Setting	λ_1	λ_2
$\vec{\lambda}_{CV}^{820}$	2^{-10}	2^{-3}
$\vec{\lambda}_{CV,2}^{820}$	2^{-10}	2^{-1}
$\vec{\lambda}_{CV}^{410}$	2^{-10}	2^{-2}

1. for each fixed value of $\vec{\lambda}$, SPR can be solved quickly
2. and that many values of $\vec{\lambda}$ can be evaluated at once.

To address the first issue, we implemented a fast numerical algorithm for solving SPR for a fixed value of $\vec{\lambda}$. See Appendix A for a derivation.

To address the second issue, our cross-validation code takes advantage of the cloud computing service PiCloud⁷ to perform the computations in parallel.

The resulting regularization parameters learned by cross-validation are summarized in Table 3.

5 The Performance of SPR

Our ultimate goal is to produce substantially better estimates of player value than least squares. If it turns out that despite all the additional computational work that SPR requires that there is little or no statistical improvement, then SPR is not of much practical value. In this section, we discuss the performance of SPR on the 2010-2011 NBA dataset. We demonstrate that SPR substantially outperforms both the dummy estimate and least squares estimate, and outperforms even Las Vegas given a sufficient amount of training data.

5.1 SPR outperforms least squares

From Table 3, we see that the cross-validation methodology described in Section 4.2 on the first 820 games of the 2010-2011 season yields the regularization parameter

$$\vec{\lambda}_{CV}^{820} = (2^{-10}, 2^{-3}).$$

Armed with this choice, we can now compare least squares to SPR on the final 410 games of the 2010-2011 NBA regular season. Each procedure produces a player rating vector $\hat{\beta}$, and we can use these ratings to predict the final margin of victory over this collection of games.

Recall that we use the variable \hat{A}_i to denote the number of points that a statistical estimator predicts that the home team will win game i , A_i to denote the *actual* number of

⁷<http://www.picloud.com>

points by which the home team wins game i , and $\hat{E}_k = \hat{A}_i - A_i$ to denote the difference between these quantities.

Figure 2 is a histogram of the variable \hat{E}_k for each technique. It is clear from Figure 2 that SPR produces better estimates than APM. The histogram of the SPR errors are “spikier” around the origin than the APM errors. We can also study some of the empirical properties of the variable \hat{E}_k for each approach. Table 2 summarizes the results. As Table 2 indicates, SPR represents a substantial improvement on APM in nearly all of these statistical measures. In particular,

- the fraction of games in which the wrong winner is guessed decreases from 33.66% with LS to 28.54% with SPR; and
- the average absolute error in predicting the margin of victory decreases from 18.05 to 10.554.

Comparing SPR to the dummy estimator, we

- see an enormous improvement in ability to predict the winning team. The percentage of games in which the wrong winner is predicted falls from 39.27% to 28.54%.
- Both techniques obtain a similar average absolute error in predicting the margin of victory, with 10.54 for the dummy estimator and 10.55 with SPR.

Overall, this suggests that SPR more accurately models the NBA than the least squares estimator.

5.2 SPR outperforms Las Vegas

To convincingly evaluate the performance of SPR, we examine whether it actually results in a profitable gambling strategy against the Vegas lines. In fact, we will compare the dummy, least squares and SPR estimators. Given predictions by each of the above estimates, we have the following natural gambling strategy:

1. If the deviation Δ between the estimate’s prediction of the outcome of a game and the Vegas lines is greater than 3, place a bet on the team the estimator favors.
2. Otherwise, don’t bet.

Due to transaction costs that the sportbooking companies charge⁸ a gambling strategy must win more than roughly 52.5% of the time to at least break even. Table 5.2 summarizes the result of this gambling rule for each of the three techniques of interest over the last 410 games of the 2010-2011 NBA season. The dummy-based gambling strategy places 263 bets on the 410 games and loses 3 more bets than it wins, for a winning percentage below 50%, which is performance comparable to random guessing, and not enough to break even.

⁸The fee is called the “vig” in the gambling community.

The least squares-based strategy has a winning percentage of 51.97% on 356 bets made. In comparison, SPR places wagers on 290 games and wins 57.24% of these bets. This represents a very profitable betting strategy, and thus suggests that SPR more accurately models the NBA than major alternatives, including the estimators used by Las Vegas. Finally, SPR obtains this improved performance while only having access to the first 820 games of the regular season.

Table 4: Betting Strategy over the last 410 games, $\Delta = 3$

Statistic	Dummy	LS	RR	SPR	SPR2
# of bets possible	410.0000	410.0000	410.0000	410.0000	410.0000
# of bets made	263.0000	356.0000	346.0000	290.0000	321.0000
Net # of bets won	-3.0000	14.0000	26.0000	42.0000	21.0000
Winning percentage	0.4943	0.5197	0.5376	0.5724	0.5327

5.3 Robustness of Results

How sensitive is the SPR algorithm to our choice of training on the first 410 games? Does the performance relative to the least squares estimate degrade if the estimators are trained on much fewer games? To evaluate this, we train estimators on the first 410 games and then evaluate predictive power on the remaining 820 games. From Table 3, we obtain the cross-validation selected regularization parameter

$$\vec{\lambda}_{CV}^{410} = (2^{-7}, 2^{-2}).$$

We also compare against the Las Vegas predictions for that block of 820 games. Table 5 summarizes the results of this experiment. As before, SPR outperforms both the Dummy estimator and LS. Furthermore, by increasing Δ to 5 (from the value 3 used when training on 820 games), SPR still leads to a successful betting strategy, as Table 6 shows.

6 What does SPR say about the NBA?

In the previous section, we evaluated the performance of SPR by testing its ability to predict the outcome of unseen games. In this section, we interpret the box score weights vector $\hat{\mathbf{z}}^{\vec{\lambda}}$ and player rating vector $\hat{\beta}^{\vec{\lambda}}$ returned by SPR, and discuss what they say about the NBA.

6.1 Top 10 players in the league

From $\hat{\beta}^{\vec{\lambda}}$ we can extract a list of the top 10 players in the league who have played at least 10 possessions. Table 7 summarizes these results. This list contains some of the most

Table 5: Robustness Experiment, First 410 Games

Metric	Dummy	LS	RR	SPR
RMSE (millions)	1087.4967	1209.3042	1130.7781	1078.7885
Fraction of games guessed wrong	0.4024	0.4073	0.3732	0.3049
Mean of $ \hat{E}_i $	10.3326	28.9714	19.9875	11.4719
Variance of $ \hat{E}_i $	64.9664	524.1851	238.1755	74.1957
Median of $ \hat{E}_i $	9.2783	23.3688	17.3077	9.5853
Min of $ \hat{E}_i $	0.0279	0.0491	0.0158	0.0208
Max of $ \hat{E}_i $	49.2299	150.4097	98.2374	43.1902
Empirical $\mathbb{P}(\hat{E}_i > 1)$	0.9207	0.9756	0.9659	0.9573
Empirical $\mathbb{P}(\hat{E}_i > 3)$	0.7768	0.9329	0.9000	0.8378
Empirical $\mathbb{P}(\hat{E}_i > 5)$	0.6744	0.8817	0.8195	0.7378
Empirical $\mathbb{P}(\hat{E}_i > 10)$	0.4720	0.7805	0.6890	0.4780

Table 6: Betting Strategy over the last 820 games, $\Delta = 5$

Statistic	Dummy	LS	RR	SPR
# of bets possible	820.0000	820.0000	820.0000	820.0000
# of bets made	342.0000	700.0000	658.0000	503.0000
Net # of bets won	-10.0000	6.0000	14.0000	57.0000
Winning percentage	0.4854	0.5043	0.5106	0.5567

Table 7. SPR Player Ratings

Player	$\hat{\beta}_i^{\bar{\lambda}}$
James, LeBron	8.6006
Garnett, Kevin	8.2860
Paul, Chris	8.1899
Nowitzki, Dirk	7.7296
Howard, Dwight	7.4706
Gasol, Pau	6.9251
Odom, Lamar	6.5630
Hilario, Nene	6.2170
Evans, Jeremy	6.1725
Nash, Steve	6.0349

prominent star players in the league (LeBron James, Chris Paul, Dirk Nowitzki, Dwight Howard), thus agreeing with common basketball wisdom. However, this ranking contradicts

common basketball wisdom in the following ways:

1. The list noticeably omits Kobe Bryant, a player pop culture and common basketball wisdom considers one of the league’s superstars. Yet SPR thinks very highly of Pau Gasol and Lamar Odom, two of Kobe Bryant’s teammates who are individually credited far less for the success of the Lakers than Kobe is.
2. The list includes Nene Hilario and Jeremy Evans, players who are not considered by most to be amongst the top 10 players in the league.

6.2 Top 10 most underrated and overrated players

There are certain players in the NBA for whom their impact on the game seems to be far more (or less) than their raw box score production suggests. SPR allows us to identify these players and quantify their impact by measuring the discrepancy between their SPR rating and their weighted box score ratings $\hat{\theta}^{\vec{\lambda}}_i$.

We define the underrated vector \mathbf{U} as

$$\mathbf{U} := \hat{\beta}^{\vec{\lambda}} - \hat{\theta}^{\vec{\lambda}}.$$

Similarly, we can examine which players impact the game much less than their box score production suggests with the vector $\mathbf{O} := -\mathbf{U}$.

Table 9 lists the top 10 most underrated/overrated players in the league relative to their box score production. For at least a few of these players, it is easy to understand why box scores alone do a poor job of capturing their impact:

- Andris Biedrens is a severe liability offensively, due to both his inability to score outside of 5 feet of the basket and poor free throw shooting. This makes it much more difficult for his teammates to score, since his defender can shift attention away from him and instead provide help elsewhere. Biedrens is also a liability defensively.
- Goran Dragic is a point guard with a scoring mentality. While a “shoot-first” point guard is not necessarily harmful to a team, if he doesn’t do a good enough job in setting up his teammates and creating easy scoring opportunities for them, it hurts his team’s ability to score.

6.3 Box score weights produced by SPR

The SPR regression also produces box score weights $\hat{\mathbf{z}}^{\vec{\lambda}}$ that tell us the relative importance of the different box score statistics. $\hat{\mathbf{z}}^{\vec{\lambda}}$ gives us a method to linearly transform box score data into the player effectiveness rating $\hat{\theta}^{\vec{\lambda}}$ defined in Equation 6. For player j , the variable $\hat{\theta}^{\vec{\lambda}}_j$ is a weighted linear combination of his box score statistics.

We can examine each entry of the vector $\hat{\mathbf{z}}^{\vec{\lambda}}$ to compare the relative importance of different box score variables like rebounds, assists and steals. Table 8 summarizes the results. We also display the relevant row in the box score matrix \mathbf{R} for LeBron James, which we call $\mathbf{R}_{\text{LeBron James}}^T$.

Examining this table, we see that LeBron James made two point shots at a rate of 7.83 per 36 minutes, and attempted two point shots at a rate of 14.16 per 36 minutes. The corresponding weightings from $\hat{\mathbf{z}}^{\vec{\lambda}, \text{rescaled}}$ are 3.38 and -1.54 respectively, suggesting that overall LeBron's rating from his two point shooting is $7.83 \times 3.38 + 14.16 \times -1.54 \approx 4.66$ points. In fact, from these weightings we can calculate that according to the SPR model all players in the league must hit their two point shots roughly 45% of the time for their rating from two point shooting to be non-negative.

Interestingly enough, a similar calculation reveals that three point shots must only be hit at a roughly 14% rate to break even. This is counterintuitive: naively one would believe that hitting two point shots q percent of the time should be equivalent to hitting three point shots $\frac{2}{3}q$ of the time. However, three point shooting increases the amount of spacing on the floor and perhaps missed three point shots are easier to rebound for the offensive team.

According to this interpretation of the $\hat{\mathbf{z}}^{\vec{\lambda}}$ variable turnovers are extremely costly, with the corresponding entry of $\hat{\mathbf{z}}^{\vec{\lambda}, \text{rescaled}}$ equal to -0.76 . Thus, LeBron's turnover rate of 3.34 turnovers per 36 minutes hurts his rating box score rating by roughly 6.28 points.

7 Extending SPR by augmenting the box score

In this section, we discuss a possible extension to the SPR model.

The box score matrix \mathbf{R} keeps track of statistics like rebounds, assists, and steals. However, one might imagine augmenting this basic box score matrix with products of raw statistics such as rebounds \times assists, blocks \times steals, turnovers \times free throws made, etc. By capturing some of these product statistics and incorporating them into SPR, one might more accurately model the value of multifaceted players.

We expand the matrix \mathbf{R} to include all pairwise product of the basic variables. If \mathbf{R} is a p by d matrix, this leads to a p by $d + \binom{d}{2}$ matrix called

$$\text{Poly}(\mathbf{R}, 2).$$

Let us use the notations $\text{SPR}(\mathbf{R})$ and $\text{SPR}(\mathbf{R}, 2)$ to denote SPR with the box score matrices \mathbf{R} and $\text{Poly}(\mathbf{R}, 2)$, respectively. Applying the cross-validation procedure described in Section 4.2 on the first 820 games of the 2010-2011 produces the regularization parameter

$$\vec{\lambda}_{CV,2}^{820} = (2^{-10}, 2^{-1}).$$

With this choice of parameter for the expanded box score matrix $\text{Poly}(\mathbf{R}, 2)$, we can then empirically compare its performance to that of the ordinary SPR algorithm using the

Table 8: Box Score Weights

Statistic	Description	$\hat{\mathbf{z}}^{\bar{\lambda}}$	$\mathbf{R}_{\text{LeBron James}}^T$
2M	Per 36 Minute	3.38	7.83
2A	Per 36 Minute	-1.54	14.16
3M	Per 36 Minute	1.48	1.08
3A	Per 36 Minute	-0.21	3.28
FTM	Per 36 Minute	0.73	5.91
FTA	Per 36 Minute	-0.33	7.79
OR	Per 36 Minute	0.11	0.94
DR	Per 36 Minute	0.50	5.99
AS	Per 36 Minute	0.85	6.51
ST	Per 36 Minute	1.66	1.46
TO	Per 36 Minute	-1.88	3.34
BK	Per 36 Minute	0.86	0.58
PF	Per 36 Minute	-0.37	1.92
TC	Per 36 Minute	2.81	0.09
DQ	Per 36 Minute	6.98	0.00
P1	Boolean	-0.17	0.00
P2	Boolean	-0.71	0.00
P3	Boolean	0.29	1.00
P4	Boolean	1.65	0.00

basic box score matrix \mathbf{R} . Figure 2 demonstrates the result of this experiment. From this figure we see that the additional box score statistics don’t seem to substantially improve performance. The histogram of the $\text{SPR}(\mathbf{R}, 2)$ errors are fairly similar to the $\text{SPR}(\mathbf{R})$ errors. We can also study some of the empirical properties of the variable \hat{E}_k for each approach. Table 2 summarizes the results.

As Table 2 indicates, $\text{SPR}(\mathbf{R}, 2)$ doesn’t improve upon the predictive power of $\text{SPR}(\mathbf{R})$. The fraction of games in which the wrong winner is guessed actually increases from 28.54% to 29.51%, the average absolute error in predicting games increases from 10.55 to 11.81.

A possible explanation for this poor statistical performance is that the pairwise interaction terms that $\text{SPR}(\mathbf{R}, 2)$ models are too many, and thus the model is overfitting.

8 Conclusion

We have introduced SPR, a powerful new statistical inference procedure for the NBA. We compared the statistical performance of our approach to an existing popular technique based on least squares and demonstrate empirically that SPR gives more predictive power. We also compare SPR to the Las Vegas lines and show that with sufficient training data, SPR seems to better predict the NBA than Vegas. We interpret the estimates produced by SPR and

Table 9: Underrated/Overrated Players

Player	$\hat{\beta}^{\vec{\lambda}}$	$\mathbf{R}\hat{\mathbf{z}}^{\vec{\lambda}} + \hat{\beta}_0^{\vec{\lambda}}\mathbf{1}_p$	Underrated
Dooling, Keyon	1.3531	-0.3840	1.7371
Watson, Earl	0.9973	-0.2920	1.2893
Aldridge, LaMarcus	5.0226	3.8456	1.1770
Ginobili, Manu	5.4336	4.2599	1.1738
Tolliver, Anthony	1.9424	0.7742	1.1682
Bosh, Chris	4.6311	3.4681	1.1630
Carter, Vince	1.4315	0.3376	1.0939
Collins, Jason	-2.4071	-3.4634	1.0563
Hill, George	2.0535	1.0201	1.0333
Bass, Brandon	2.5777	1.5792	0.9985

Player	$\hat{\beta}^{\vec{\lambda}}$	$\mathbf{R}\hat{\mathbf{z}}^{\vec{\lambda}} + \hat{\beta}_0^{\vec{\lambda}}\mathbf{1}_p$	Overrated
Dragic, Goran	-2.1439	-0.4758	-1.6680
Marion, Shawn	0.7049	2.2120	-1.5071
Gortat, Marcin	2.7433	4.1681	-1.4249
Ellis, Monta	-0.1175	1.2634	-1.3810
Biedrins, Andris	0.9205	2.2408	-1.3203
Jefferson, Al	1.7706	3.0598	-1.2893
Felton, Raymond	1.4721	2.7513	-1.2792
Bell, Raja	-2.8779	-1.5992	-1.2787
Dudley, Jared	1.1287	2.3858	-1.2572
Law, Acie	-1.7759	-0.6247	-1.1512

discuss what they suggest about who the best players in the NBA are, and which players are overrated or underrated. Finally, we discuss a possible extension to the SPR model.

A The Cyclical Coordinate Descent Algorithm for SPR

There are a variety of techniques for solving the convex program (5), including interior-point methods (Boyd and Vandenberghe, 2004), LARs (Efron et al., 2004), iteratively re-weighted least squares (Huber, 1974), approximating the ℓ_1 term with a smooth function (Lee et al., 2006) the sub-gradient method (Shor et al., 1985), and Nesterov’s proximal gradient method (Nesterov, 2003).

Ultimately, we found experimentally that cyclical coordinate descent (CCD) (Friedman et al., 2007; Wu and Lange, 2008) was the fastest for our problem.

The CCD method works by repeatedly optimizing the objective function viewed as a function of each variable with the others fixed. This idea gives a CCD algorithm for SPR, Algorithm 1.

Algorithm 1 CCDSPPR($\mathbf{X}, \mathbf{Y}, \mathbf{R}, \vec{\lambda}, T$)

```

1:  $\alpha_{\text{hca}}(0) \leftarrow 0, z_0(0) \leftarrow 0, \beta(0) \leftarrow \mathbf{0}_p, \mathbf{z}(0) \leftarrow \mathbf{0}_d$ 
2: for  $i \in \{1, 2, \dots, T\}$  do
3:   {Optimize  $\alpha_{\text{hca}}$  with all other variables fixed}
4:   {Optimize  $z_0$  with all other variables fixed}
5:   for  $k \in \{1, 2, \dots, p\}$  do
6:     {Optimize  $\beta_k$  with all other variables fixed}
7:   end for
8:   for  $\ell \in \{1, 2, \dots, d\}$  do
9:     {Optimize  $\mathbf{z}_\ell$  with all other variables fixed}
10:  end for
11: end for
12: return  $\alpha_{\text{hca}}(T), z_0(T), \beta(T), \mathbf{z}(T)$ 

```

A.1 Convergence of Algorithm 1

The correctness of this algorithm for minimizing the objective function (4) follows from Lemma A.1.

Lemma A.1. *Let $\alpha_{\text{hca}}^*, z_0^*, \beta^*, \mathbf{z}^* \in \arg \min g(\alpha_{\text{hca}}, \beta, z_0, \mathbf{z}; \vec{\lambda})$.*

Then

$$\lim_{T \rightarrow \infty} g(\alpha_{\text{hca}}(T), z_0(T), \beta(T), \mathbf{z}(T)) = g(\alpha_{\text{hca}}^*, z_0^*, \beta^*, \mathbf{z}^*).$$

Furthermore, when $\alpha_{\text{hca}}^, z_0^*, \beta^*, \mathbf{z}^*$ is the unique global minimum of g ,*

$$\lim_{T \rightarrow \infty} (\alpha_{\text{hca}}(T), z_0(T), \beta(T), \mathbf{z}(T)) = (\alpha_{\text{hca}}^*, z_0^*, \beta^*, \mathbf{z}^*).$$

Proof. This is a direct consequence of Proposition 5.1 of Tseng (2001). In particular, identify f_0 and $f_i, i = 1, \dots, p$ of Proposition 5.1 with $L_{\text{quadratic}}(\alpha_{\text{hca}}, \boldsymbol{\beta}) + \lambda_2 \|\boldsymbol{\beta} - z_0 \mathbf{1}_p - \mathbf{R}\mathbf{z}\|_2^2$ and $\lambda_1 |\boldsymbol{\beta}_i|, i = 1, \dots, p$, respectively. We observe that

- Assumption B1 of Tseng (2001) is satisfied, since f_0 is continuous.
- Assumption B2 of Tseng (2001) is satisfied, since f is convex and non-constant on line segments.
- Assumption B3 is satisfied, since $f_i, i = 1, \dots, p$ are continuous.
- Assumption C2 is trivially satisfied.

Therefore, the conditions of Proposition 5.1 of Tseng (2001) are satisfied for Algorithm 1 on the objective function (3).

Since (4) has at least one global minimum and is convex, then we further conclude that the limit points of Algorithm 1 are global minima. □

A.2 Computing the updates for Algorithm 1

The updates for $\alpha_{\text{hca}}(i), z_0(i), \boldsymbol{\beta}(i), \mathbf{z}(i)$ can be computed in closed form.

To compute $\alpha_{\text{hca}}(i)$, we can optimize the objective function g viewed as a function only of the decision variable α_{hca} by taking the derivative and setting it to zero.

This yields the update

$$\alpha_{\text{hca}} \leftarrow \frac{\mathbf{1}_n^T \text{Diag}(w) [\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}]}{\mathbf{1}_n^T w}.$$

Similarly, for $z_0(i)$ we get the update

$$z_0 \leftarrow \frac{1}{p} \mathbf{1}_p^T [\boldsymbol{\beta} - \mathbf{R}\mathbf{z}].$$

For \mathbf{z} , we simply get the least squares updates:

$$\mathbf{z} \leftarrow (\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T [\boldsymbol{\beta} - z_0 \mathbf{1}_p].$$

A.2.1 Updates for $\boldsymbol{\beta}$

We next derive a closed-form expression for the updates for $\boldsymbol{\beta}_i$. To do so, we need Lemma A.2.

Lemma A.2 (One-variable lasso is soft-thresholding). *Let $h(x) := \frac{1}{2}Ax^2 - Bx + C + \tau|x|$, $x \in \mathbb{R}$. Suppose that $A > 0$. The solution of*

$$\min_{x \in \mathbb{R}} h(x) \tag{7}$$

is

$$x^* = \frac{S_\tau(B)}{A}$$

where

$$S_\tau(x) := \begin{cases} 0 & \text{if } |x| \leq \tau \\ x - \tau & \text{if } x > 0 \text{ and } |x| > \tau \\ x + \tau & \text{if } x < 0 \text{ and } |x| > \tau, \end{cases}$$

is the soft-thresholding function with threshold τ .

See Section A.2.2 for a proof of this.

This lemma is useful, as it allows us to immediately write the updates for $\beta_k(i)$.

First, let us identify A and B for $\beta_k(i)$. Differentiating g_s , we get

$$\begin{aligned} \partial_{\beta_t} g_s &= \partial_{\beta_t} (L_{\text{quadratic}}(\alpha_{\text{hca}}, \beta) + \lambda_2 \|\beta - z_0 \mathbf{1}_p - \mathbf{R}\mathbf{z}\|_2^2) \\ &= \partial_{\beta_t} \left[\frac{1}{\mathbf{1}_n^T \vec{w}} \sum_i w_i (Y_i - \alpha_{\text{hca}} - \mathbf{X}_i^T \beta)^2 \right] + \lambda_2 \partial_{\beta_t} \sum_j (\beta_j - z_0 - \mathbf{R}_j^T z)^2 \\ &= \left(\frac{1}{\mathbf{1}_n^T \vec{w}} \sum_i w_i \partial_{\beta_t} (Y_i - \alpha_{\text{hca}} - \mathbf{X}_i^T \beta)^2 \right) + \lambda_2 \sum_j \partial_{\beta_t} (\beta_j - z_0 - \mathbf{R}_j^T z)^2 \\ &= \frac{1}{\mathbf{1}_n^T \vec{w}} \sum_i 2w_i \mathbf{X}_{it} (-Y_i + \alpha_{\text{hca}} + \mathbf{X}_i^T \beta) + \lambda_2 2(\beta_t - z_0 - \mathbf{R}_t^T z) \\ &= C \sum_i w_i \mathbf{X}_{it} (-Y_i + \alpha_{\text{hca}} + \mathbf{X}_i^T \beta) + \lambda_2 2(\beta_t - z_0 - \mathbf{R}_t^T z) \\ &= C(\mathbf{X}e_t)^T W(-Y + \alpha_{\text{hca}} \mathbf{1}_n + \mathbf{X}\beta) + \lambda_2 2(\beta_t - z_0 - \mathbf{R}_t^T z) \\ &= C(\mathbf{X}e_t)^T W(-Y + \alpha_{\text{hca}} \mathbf{1}_n + \mathbf{X}[\beta - e_t \beta_t + e_t \beta_t]) + \lambda_2 2(\beta_t - \theta_t) \\ &= C(\mathbf{X}e_t)^T W(\kappa + \mathbf{X}[e_t \beta_t]) + \lambda_2 2(\beta_t - \theta_t) \end{aligned}$$

where

$$\begin{aligned} C &:= \frac{2}{\mathbf{1}_n^T \vec{w}}, \\ \theta_t &:= (z_0 \mathbf{1}_p + \mathbf{R}\mathbf{z})^T e_t, \\ \kappa &:= -Y + \alpha_{\text{hca}} \mathbf{1}_n + \mathbf{X}[\beta - e_t \beta_t]. \end{aligned}$$

The constant term (with respect to β_t) of the above expression is

$$D := C(\mathbf{X}e_t)^T W \kappa - 2\lambda_2 \theta_t.$$

The linear term is

$$\begin{aligned} C(\mathbf{X}e_t)^T W \mathbf{X}e_t \beta_t + 2\lambda_2 \beta_t &= [Ce_t^T \mathbf{X}^T W \mathbf{X}e_t + 2\lambda_2] \beta_t \\ &= E \beta_t, \end{aligned}$$

where

$$E := Ce_t^T \mathbf{X}^T W \mathbf{X}e_t + 2\lambda_2.$$

From this we conclude that for $\beta_k(t)$

$$\begin{aligned} A &= E \\ B &:= -D. \end{aligned}$$

So, we have the update equation

$$\beta_k(i) \leftarrow \frac{S_{\lambda_1}(B)}{A}.$$

A.2.2 Proof of Lemma A.2

Proof. The subdifferential of $h(x)$ (Rockafellar, 1970) is the set

$$\begin{aligned} \partial h(x) &:= \sum_{k=1}^K a_k(a_k x - b_k) + \tau \partial |x| \\ &= xA - B + \tau \partial |x|, \end{aligned}$$

where

$$\begin{aligned} A &:= \sum_{k=1}^K a_k^2 \\ B &:= \sum_{k=1}^K b_k \\ \partial |x| &:= \begin{cases} \{\text{sign}(x)\} & \text{if } x \neq 0 \\ [-1, 1] & \text{otherwise.} \end{cases} \end{aligned}$$

From the theory of convex analysis (Rockafellar, 1970) x^* is the solution of (7) if and only if

$$0 \in \partial h(x^*). \tag{8}$$

The set $\partial h(x^*)$ behaves differently depending on the value of x^* . When $x^* \neq 0$, then

$$0 \in \partial h(x^*) = \{x^* A - B + \tau \operatorname{sign}(x^*)\},$$

which is equivalent to $x^* = \frac{B - \tau \operatorname{sign}(x^*)}{A}$. However, when $x^* = 0$, then

$$0 \in \partial h(x^*) = \{x^* A - B + \tau[-1, 1]\}.$$

We use this observation to deal with the following two cases:

1. Suppose that $\tau \geq |B|$. Then

(a) If $x^* \neq 0$, then

$$\begin{aligned} \tau &\geq |B| = |x^* A + \tau \operatorname{sign}(x^*)| \\ &= x^* A + \tau, \end{aligned}$$

since at least one $a_k \neq 0$, then $A > 0$. This is a contradiction. Therefore $x^* \neq 0$ cannot be a solution when $\tau \geq |B|$.

(b) If $x^* = 0$, then

$$0 \in -B + \tau[-1, 1] = [-\tau - B, \tau - B],$$

which is true.

2. Suppose that $\tau < |B|$. Then

(a) If $x^* \neq 0$, then $B - \tau \operatorname{sign}(x)$ has the same sign as B . Since A is positive, then x^* has the same sign as B . So the choice of $x^* = \frac{B - \tau \operatorname{sign}(B)}{A}$ satisfies the required sub-gradient optimality condition (8) without contradiction.

(b) If $x^* = 0$, then $0 \in -B + \tau[-1, 1] = [-\tau - B, \tau - B]$, which is a contradiction.

Thus,

1. $\tau \geq |B| \implies x^* = 0$,
2. $\tau < |B| \implies x^* = \frac{S_\tau(B)}{A}$.

These two cases can be summarized by $x^* = \frac{S_\tau(B)}{A}$, as desired. \square

References

- S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, UK, 2004.
- B. Efron, T. Hastie, I. Johnstone, and R. Tibshirani. Least angle regression. *Annals of Statistics*, 32(2):407–499, 2004.
- Jerome Friedman, Trevor Hastie, Holger Höfling, and Robert Tibshirani. Pathwise coordinate optimization. *Annals of Applied Statistics*, 2:302–332, 2007.
- Peter Huber. *Robust Statistics*. Wiley, New York, 1974.
- Justin Kubatko, Dean Oliver, Kevin Pelton, and Dan Rosenbaum. A starting point for analyzing basketball statistics. *Journal of Quantitative Analysis in Sports*, 2007.
- Su I. Lee, Honglak Lee, Pieter Abbeel, and Andrew Y. Ng. Efficient L1 Regularized Logistic Regression. In *Proceedings of the Twenty-First National Conference on Artificial Intelligence (AAAI-06)*. 2006.
- Yurii Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course (Applied Optimization)*. Springer Netherlands, 1 edition, 2003.
- Dean Oliver. *Basketball on Paper: Rules and Tools for Performance Analysis*. Potomac Books, 2004.
- G. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.
- N. Z. Shor, K. C. Kiwiel, and A. Ruszcaynski. *Minimization methods for non-differentiable functions*. Springer-Verlag New York, Inc., New York, NY, USA, 1985.
- M. Stone. Cross-validatory choice and assessment of statistical predictions. *Journal of the Royal Statistical Society. Series B (Methodological)*, 36(2):pp. 111–147, 1974.
- P. Tseng. Convergence of Block Coordinate Descent Method for Nondifferentiable Minimization. *J. Optim. Theory Appl.*, 109:475–494, 2001.
- Tong Tong Wu and Kenneth Lange. Coordinate descent algorithms for lasso penalized regression. *Annals of Applied Statistics*, 2:224–244, 2008.